Symbolic Computation and Non-travelling Wave Solutions of the (2+1)-Dimensional Korteweg de Vries Equation

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In this paper, with the aid of symbolic computation we improve the extended F-expansion method described in Chaos, Solitons and Fractals **22**, 111 (2004) to solve the $(2+1)$-dimensional Korteweg de Vries equation. Using this method, we derive many exact non-travelling wave solutions. These are more general than the previous solutions derived with the extended F-expansion method. They include the Jacobi elliptic function, soliton-like trigonometric function solutions, and so on. Our method can be applied to other nonlinear evolution equations.

Key words: Non-travelling Wave Solution; $(2+1)$-Dimentional Korteweg de Vries Equation; Jacobi Elliptic Function; Soliton-like.

1. Introduction

In recent years, nonlinear evolution equations have attracted considerable attention. Many powerful methods have been proposed to find exact solutions of nonlinear evolution equations (NEEs). For example, in the past decades there has been significant progress in the development of methods such as the inverse scattering [1], the Bäcklund transformation [2 – 5], the Darboux transformation [6 – 8], the Hirota bilinear [9 – 10], the algebro-geometric [11 – 12] and the tanh-function [13], variable separation [14].

Korteweg de Vries (KdV) and KdV-type equations are found in many apparently unrelated phenomena such as in plasmas and fluids, and in lattice vibrations of crystals at low temperatures. All these applications start from a more or less general physical model and end up in the KdV equation by considering a specific limit of the physical problem. In this sense the KdV equation is universal. The balance of the dispersion of linear waves with the non-linearity stabilizes the solution of the equation and results in the outstanding behavior of regularity. The $(2+1)$-dimensional KdV equation

$$\left.\begin{aligned} &u_t + u_{xxx} - 3uv_x - 3vu_x = 0,\\ &u_x = v_y \end{aligned}\right\} \qquad (1)$$

was first derived by Boiti *et al.* [15], using the idea of the weak Lax pair. Subsequently, many people analyzed its localized structure and presented a series of methods to get its exact solutions. For example, Radha and Lakshmanan [16] analyzed the localized coherent structures of the $(2+1)$-dimensional KdV equation. Lou and Ruan [17] revised the localized excitations of the $(2+1)$-dimensional KdV equation. Elwakil *et al.* [18] and Xuan *et al.* [19] found further exact solutions using homogeneous balance and the generalized Riccati equation expansion method, respectively.

In this paper, with the aid of symbolic computation we improve the extended F-expansion method in [20] to solve the $(2+1)$-dimensional Korteweg de Vries equation. Our method is more convenient and general than the method in [21]. If we let the modulus of the Jacobi elliptic function approach 1 or 0, we find additional solutions to (1). The properties of exact solutions of the KdV equation are shown as examples in some figures. We show that the solutions we get are more general than those which the extended F-expansion method yields. In fact, our method is also powerful to solve other nonlinear evolution equations.

The paper is organized as follows: in Section 2, we derive our method; in Section 3, we apply it to the (2+1)-dimensional KdV equation. We give conclusions in the last section.

2. Summary of our Method

In this section, based on symbolic computation, we describe our method.

Consider a given NEE with independent variables t, x, y, and a physical field $u = u(t, x, y)$:

$$W(u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, \cdots\cdots) = 0. \tag{2}$$

We solve (2) with the ansatz

$$u(x) = a_0 + \sum_{i=1}^{n} f^{i-1}(\omega)(a_i f(\omega) + b_i g(\omega)), \; a_i^2 + b_i^2 \neq 0, \tag{3}$$

where the functions $f(\omega)$ and $g(\omega)$ satisfy the relations

$$\left.\begin{aligned} f'^2(\omega) &= l_1 f^4(\omega) + m_1 f^2(\omega) + n_1, \\ g'^2(\omega) &= l_2 g^4(\omega) + m_2 g^2(\omega) + n_2, \\ g^2(\omega) &= \frac{l_1 f^2(\omega)}{l_2} + \frac{m_1 - m_2}{3 l_2}, \\ n_1 &= \frac{m_1^2 - {m_2}^2 + 3 l_2 n_2}{3 l_1}, \end{aligned}\right\} \tag{4}$$

where $f' = \frac{\mathrm{d}}{\mathrm{d}\omega} f(\omega)$, $g' = \frac{\mathrm{d}}{\mathrm{d}\omega} g(\omega)$, $\omega = \omega(t, x, y)$, and $a_0, a_i, b_i \, (i = 1, 2, \cdots, n)$ are undetermined functions of (t, x, y). n is an integer which is determined by balancing the highest order derivative term with the highest order nonlinear term in (2).

The ansatz (3) is more general than previous ones, because it contains two independent functions $f(\omega)$ and $g(\omega)$ and the coefficients $a_0, a_i, b_i \, (i = 1, 2, \cdots, n)$ are allowed to be functions of (x, y, t) instead of being constants.

Substituting (3) into the given NEE (2) with (4) and collecting the coefficients of the polynomials of $f(\omega)$, $g(\omega)$ and $\sqrt{l_1 f^4(\omega) + m_1 f^2(\omega) + n_1}$, then setting each coefficient to zero, we can deduce a set of over-determined partial differential equations. They are solved with Maple, determining the values of a_0, a_i, b_i, ω $(i = 1, 2, \cdots, n)$. At the same time we select proper values of parameters l_1, m_1, n_1, l_2, m_2, n_2 to determine $f(\omega)$, $g(\omega)$. Finally, substituting $a_0, a_i, b_i (i = 1, 2, \cdots, n)$ and ω into (3) with the corresponding solutions of $f(\omega), g(\omega)$, we derive exact solutions of the given NEE (2).

As is known, when $k \to 1$, the Jacobi elliptic functions degenerate to hyperbolic functions, and when $k \to 0$, the Jacobi elliptic functions degenerate to trigonometric functions. So, by this method we can get many other exact solutions of NEEs.

Here, it is necessary to point out that the ansatz

$$u(\omega) = \sum_{j=0}^{n} \sum_{i=0}^{j} c_{ji} F^i(\omega) G^{j-i}(\omega) \tag{5}$$

in [1] is less general because $F(\omega)$ and $G(\omega)$ are related by

$$G^2(\omega) = \frac{l_1 F^2(\omega)}{l_2} + \frac{m_1 - m_2}{3 l_2}. \tag{6}$$

For example, if $n = 2$ the ansatz (5) becomes

$$\begin{aligned} u(\omega) = {} & c_{00} + c_{10} G(\omega) + c_{11} F(\omega) \\ & + c_{20} G^2(\omega) + c_{21} F(\omega) G(\omega) + c_{22} F^2(\omega). \end{aligned} \tag{7}$$

Because of (6), $G^2(\omega)$ is redundant. We therefore use the more general transform (3).

Remark 1. We point out that for the more general ansatz (3), a more complicated computation is expected than before, even if the symbolic computer systems like Maple allows us to perform the complicated and tedious algebraic and differential calculations on a computer. In general it is very difficult, sometime impossible, to solve the set of over-determined partial differential equations. As the calculation goes on, in order to drastically simplify the work or make it feasible, we often choose special functions for $a_0, a_i, b_i \, (i = 1, 2, \cdots, n)$, and ω.

We may extend the above method in the following way: We rewrite the ansatz (3) to be

$$\begin{aligned} & u(x) = \sum_{i=-n}^{n} f^{i-1}(\omega)(a_i f(\omega) + b_i g(\omega)), \\ & a_i^2 + b_i^2 \neq 0. \end{aligned} \tag{8}$$

Then, using this ansatz, we can find many explicit solutions of (2) and we will consider this ansatz in another paper.

3. Non-travelling Wave Solutions of the $(2+1)$-Dimentional KdV Equation

In this section, we construct solutions of (1), using the method described in Section 2.

By the balancing procedure we get $n = 2$. Because there are two dependent variables, the ansatz (3) reads as follows:

$$\begin{cases} u = a_0 + a_1 f(\alpha x + \xi) + b_1 g(\alpha x + \xi) \\ \qquad + a_2 f^2(\alpha x + \xi) + b_2 f(\alpha x + \xi) g(\alpha x + \xi), \\ v = c_0 + c_1 f(\alpha x + \xi) + d_1 g(\alpha x + \xi) \\ \qquad + c_2 f^2(\alpha x + \xi) + d_2 f(\alpha x + \xi) g(\alpha x + \xi), \end{cases} \tag{9}$$

where α is a nonzero constant to be determined and $\xi, a_0, a_1, b_1, c_0, c_1, d_1, a_2, b_2, c_2, d_2$ are all functions of (y,t), also to be determined. Here we assume $\xi = p+q$ (with $p = p(y), q = q(t)$) for convenience.

Substituting (9) into (3) with (4) and collecting the coefficients of the polynomials of $f(\alpha x+p+q)$, $g(\alpha x+p+q)$, $\{l_1 f^4(\alpha x+p+q)+m_1 f^2(\alpha x+p+q)+n_1\}^{1/2}$, then setting each coefficient to zero, we can deduce a set of over-determined partial differential equations about $\alpha, p, q, a_0, a_1, b_1, c_0, c_1, d_1, a_2, b_2, c_2, d_2$ as follows:

$$\begin{aligned}
&-6d_2l_1p_y+6b_2l_1\alpha=0, \quad 72b_2\alpha^3{l_1}^2-36\alpha c_2b_2l_1-36\alpha a_2d_2l_1=0,\\
&6a_2\alpha l_2-6c_2p_yl_2=0, \quad b_2\alpha m_1-b_2\alpha m_2-d_2p_ym_1+d_2p_ym_2=0,\\
&18b_1l_1^2\alpha^3-27\alpha c_2b_1l_1-27\alpha c_1b_2l_1-27\alpha a_1d_2l_1-27\alpha a_2d_1l_1=0,\\
&-27\alpha a_1c_2l_2+18a_1\alpha^3l_2l_1-27\alpha b_1d_2l_1-27\alpha a_2c_1l_2-27\alpha b_2d_1l_1=0,
\end{aligned} \tag{10}$$

Because there are so many over-determined partial differential equations, only part of them is shown here for convenience.

Solving the over-determined partial differential equations (10) by using the PDE tools package of the computerized symbolic computation system Maple, we derive several solutions as follows:

Case 1.

$$\begin{aligned}
&b_2=F_1(y), \quad b_1=0, \quad a_1=0, \quad d_1=0, \quad c_1=0, \quad a_0=C_2F_1(y), \quad c_0=F_2(t), \quad d_2=C_1,\\
&c_2=\frac{\sqrt{l_2l_1}C_1}{l_2}, \quad a_2=\frac{\sqrt{l_2l_1}F_1(y)}{l_2}, \quad p=\int -\frac{F_1(y)\sqrt{l_2l_1\sqrt{l_2l_1}C_1}}{l_2l_1C_1}\mathrm{d}y+C_5, \quad \alpha=-\frac{\sqrt{l_2l_1\sqrt{l_2l_1}C_1}}{l_2l_1},\\
&q=\int \frac{\left(\sqrt{l_2l_1}C_1m_2+3\sqrt{l_2l_1}C_1m_1-3C_2C_1l_2l_1-3l_2l_1F_2(t)\right)\sqrt{l_2l_1\sqrt{l_2l_1}C_1}}{{l_2}^2{l_1}^2}\mathrm{d}t+C_6,
\end{aligned} \tag{11}$$

where $F_1(y)$ is an arbitrary function of y, $F_2(t)$ is an arbitrary function of t and C_1, C_2, C_5, C_6 are arbitrary constants.

Case 2.

$$\begin{aligned}
&b_2=F_1(y), \quad b_1=0, \quad a_1=0, \quad d_1=0, \quad c_1=0, \quad a_0=C_2F_1(y), \quad c_0=F_2(t), \quad d_2=C_1,\\
&a_2=-\frac{\sqrt{l_2l_1}F_1(y)}{l_2}, \quad c_2=-\frac{\sqrt{l_2l_1}C_1}{l_2}, \quad p=\int \frac{F_1(y)\sqrt{-l_2l_1\sqrt{l_2l_1}C_1}}{l_2l_1C_1}\mathrm{d}y+C_4, \alpha=\frac{\sqrt{-l_2l_1\sqrt{l_2l_1}C_1}}{l_2l_1},\\
&q=\int \frac{\left(\sqrt{l_2l_1}C_1m_2+3\sqrt{l_2l_1}C_1m_1+3C_2C_1l_2l_1+3l_2l_1F_2(t)\right)\sqrt{-l_2l_1\sqrt{l_2l_1}C_1}}{{l_2}^2{l_1}^2}\mathrm{d}t+C_6,
\end{aligned} \tag{12}$$

where $F_1(y)$ is an arbitrary function of y, $F_2(t)$ is an arbitrary function of t and C_1, C_2, C_4, C_6 are arbitrary constants.

Case 3.

$$\begin{aligned}
&b_1=0, \quad a_1=0, \quad d_1=0, \quad c_1=0, \quad p=\int -\frac{F_2(y)\sqrt{2}\sqrt{l_1C_1}}{2l_1C_1}\mathrm{d}y+C_3,\\
&c_2=C_1, \quad c_0=F_1(t), \quad a_0=0, \quad a_2=F_2(y), \quad \alpha=-\frac{\sqrt{2}\sqrt{l_1C_1}}{2l_1},\\
&q=\int -\frac{\sqrt{2}\sqrt{l_1C_1}(3F_1(t)l_1-2C_1m_1)}{2{l_1}^2}\mathrm{d}t+C_4, \quad b_2=0, \quad d_2=0,
\end{aligned} \tag{13}$$

where $F_2(y)$ is an arbitrary function of y, $F_1(t)$ is an arbitrary function of t and C_1, C_3, C_4 are arbitrary constants.

Case 4.

$$
\begin{aligned}
&a_2 = F_2(y), \quad \alpha = \frac{\sqrt{2}\sqrt{l_1 C_1}}{2l_1}, \quad b_1 = 0, \quad a_1 = 0, \quad d_1 = 0, \quad c_1 = 0,\\
&p = \int \frac{F_2(y)\sqrt{2}\sqrt{l_1 C_1}}{2l_1 C_1}\mathrm{d}y + C_2, \quad c_2 = C_1, \quad c_0 = F_1(t), \quad a_0 = 0, \quad b_2 = 0, \quad d_2 = 0,\\
&q = \int \frac{(3/2F_1(t)l_1 - C_1 m_1)\sqrt{l_1 C_1 \sqrt{2}}}{l_1{}^2}\mathrm{d}t + C_4, \quad b_2 = 0, \quad d_2 = 0,
\end{aligned} \tag{14}
$$

where $F_2(y)$ is an arbitrary function of y, $F_1(t)$ is an arbitrary function of t and C_1, C_2, C_4 are arbitrary constants.
With these results we get the general form (9) of solutions of (1).

Type 3.1. If we choose $l_1 = k^2$, $m_1 = -(1+k^2)$, $n_1 = 1$, $l_2 = -k^2$, $m_2 = (2k^2-1)$, $n_2 = (1-k^2)$, from (4), we derive $f = \mathrm{sn}(\alpha x + \xi)$, $g = \mathrm{cn}(\alpha x + \xi)$.

Thus we get the following Jacobi elliptic function solutions of (1):

$$
\left.\begin{aligned}
v &= c_0 + c_1\mathrm{sn}(\alpha x + \xi) + d_1\mathrm{cn}(\alpha x + \xi) + c_2\mathrm{sn}^2(\alpha x + \xi) + d_2\mathrm{sn}(\alpha x + \xi)\mathrm{cn}(\alpha x + \xi),\\
u &= a_0 + a_1\mathrm{sn}(\alpha x + \xi) + b_1\mathrm{cn}(\alpha x + \xi) + a_2\mathrm{sn}^2(\alpha x + \xi) + b_2\mathrm{sn}(\alpha x + \xi)\mathrm{cn}(\alpha x + \xi),
\end{aligned}\right\} \tag{15}
$$

where $\xi = p + q$ and α, p, q, a_0, a_1, b_1, c_0, c_1, d_1, a_2, b_2, c_2, d_2 satisfy (11)–(14), respectively.
For example, if the functions satisfy (11), namely Case 1, we derive the following Jacobi elliptic function solution of (1):

$$
\begin{aligned}
u_{11} &= C_2 F_1(y) - \sqrt{-k^4}F_1(y)\mathrm{sn}^2\left(\frac{\sqrt{-k^4\sqrt{-k^4}C_1}x}{k^4} + p + q\right)k^{-2}\\
&\quad + F_1(y)\mathrm{sn}\left(\frac{\sqrt{-k^4\sqrt{-k^4}C_1}x}{k^4} + p + q\right)\mathrm{cn}\left(\frac{\sqrt{-k^4\sqrt{-k^4}C_1}x}{k^4} + p + q\right),\\
v_{11} &= F_2(t) - \frac{\sqrt{-k^4}C_1}{k^2}\mathrm{sn}^2\left(\frac{\sqrt{-k^4\sqrt{-k^4}C_1}x}{k^4} + p + q\right)\\
&\quad + C_1\mathrm{sn}\left(\frac{\sqrt{-k^4\sqrt{-k^4}C_1}x}{k^4} + p + q\right)\mathrm{cn}\left(\frac{\sqrt{-k^4\sqrt{-k^4}C_1}x}{k^4} + p + q\right),
\end{aligned} \tag{16}
$$

where $p = \int \frac{F_1(y)\sqrt{-k^4\sqrt{-k^4}C_1}}{k^4 C_1}\mathrm{d}y + C_5$ and $q = \int \frac{\left(-\sqrt{-k^4}C_1 k^2 - 4\sqrt{-k^4}C_1 + 3C_2 C_1 k^4 + 3k^4 F_2(t)\right)\sqrt{-k^4\sqrt{-k^4}C_1}}{k^8}\mathrm{d}t + C_6$. If $k \to 1$, $\mathrm{sn}(\omega) \to \tanh(\omega)$ and $\mathrm{cn}(\omega) \to \mathrm{sech}(\omega)$, so we get a soliton-like solution from (16):

$$
\begin{aligned}
u_{12} &= C_2 F_1(y) - \mathrm{i}F_1(y)\tanh^2\left(\sqrt{-\mathrm{i}C_1}x + p + q\right) + F_1(y)\tanh\left(\sqrt{-\mathrm{i}C_1}x + p + q\right)\mathrm{sech}\left(\sqrt{-\mathrm{i}C_1}x + p + q\right),\\
v_{12} &= F_2(t) - \mathrm{i}C_1\tanh^2\left(\sqrt{-\mathrm{i}C_1}x + p + q\right) + C_1\tanh\left(\sqrt{-\mathrm{i}C_1}x + p + q\right)\mathrm{sech}\left(\sqrt{-\mathrm{i}C_1}x + p + q\right),
\end{aligned} \tag{17}
$$

where $p = \int \frac{F_1(y)\sqrt{-\mathrm{i}C_1}}{C_1}dy + C_5$ and $q = \int(-5\mathrm{i}C_1 + 3C_2C_1 + 3F_2(t))\sqrt{-\mathrm{i}C_1}\mathrm{d}t + C_6$.
So from Case 1, we find the Jacobi elliptic function, soliton-like solution of (1). We can also derive many other solutions if we make use of the other cases.

Type 3.2. If we choose $l_1 = 1$, $m_1 = (2-k^2)$, $n_1 = 1-k^2$, $l_2 = 1$, $m_2 = (2k^2-1)$, $n_2 = -k^2(1-k^2)$, from (4), we derive $f = \mathrm{cs}(\alpha x + \xi)$, $g = \mathrm{ds}(\alpha x + \xi)$.

Thus we get the following Jacobi elliptic function solutions of (1):

$$\left.\begin{aligned} u &= a_0 + a_1\mathrm{cs}(\alpha x+\xi) + b_1\mathrm{ds}(\alpha x+\xi) + a_2\mathrm{cs}^2(\alpha x+\xi) + b_2\mathrm{cs}(\alpha x+\xi)\mathrm{ds}(\alpha x+\xi),\\ v &= c_0 + c_1\mathrm{cs}(\alpha x+\xi) + d_1\mathrm{ds}(\alpha x+\xi) + c_2\mathrm{cs}^2(\alpha x+\xi) + d_2\mathrm{cs}(\alpha x+\xi)\mathrm{ds}(\alpha x+\xi),\end{aligned}\right\} \tag{18}$$

where $\xi = p+q$ and α, p, q, a_0, a_1, b_1, c_0, c_1, d_1, a_2, b_2, c_2, d_2 satisfy (11)–(14), respectively.

For example, if the functions satisfy (11), namely Case 1, we derive the following Jacobi elliptic function solution of (1):

$$\begin{aligned} u_{21} &= C_2F_1(y) + F_1(y)\mathrm{cs}^2\left(-\sqrt{C_1}x+p+q\right) + F_1(y)\mathrm{cs}\left(-\sqrt{C_1}x+p+q\right)\mathrm{ds}\left(-\sqrt{C_1}x+p+q\right),\\ v_{21} &= F_2(t) + C_1\mathrm{cs}^2\left(-\sqrt{C_1}x+p+q\right) + C_1\mathrm{cs}\left(-\sqrt{C_1}x+p+q\right)\mathrm{ds}\left(-\sqrt{C_1}x+p+q\right),\end{aligned} \tag{19}$$

where $p = \int -\frac{F_1(y)}{\sqrt{C_1}}\mathrm{d}y + C_5$ and $q = \int -(-C_1(2k^2-1) - 6C_1 + 3C_1k^2 + 3C_2C_1 + 3F_2(t))\sqrt{C_1}\mathrm{d}t + C_6$.

When $k\to 1$, $\mathrm{cs}(\omega)\to \mathrm{sech}(\omega)\coth(\omega)$ and $\mathrm{ds}(\omega)\to \mathrm{sech}(\omega)\coth(\omega)$, so we get the degenerative soliton-like solution from the solution (19):

$$u_{22} = C_2F_1(y) + 2F_1(y)\mathrm{csch}^2\left(\sqrt{C_1}x-p-q\right),\quad v_{22} = F_2(t) + 2C_1\mathrm{csch}^2\left(\sqrt{C_1}x-p-q\right), \tag{20}$$

where $p = \int -\frac{F_1(y)}{\sqrt{C_1}}\mathrm{d}y + C_5$ and $q = \int -(-4C_1 + 3C_2C_1 + 3F_2(t))\sqrt{C_1}\mathrm{d}t + C_6$.

If $k\to 0$, $\mathrm{cs}(\omega)\to\cot(\omega)$ and $\mathrm{ds}(\omega)\to\csc(\omega)$, we get the triangular function solution from the solution (19):

$$\begin{aligned} u_{23} &= C_2F_1(y) + F_1(y)\cot^2\left(-\sqrt{C_1}x+p+q\right) + F_1(y)\cot\left(-\sqrt{C_1}x+p+q\right)\csc\left(-\sqrt{C_1}x+p+q\right),\\ v_{23} &= F_2(t) + C_1\cot^2\left(\sqrt{C_1}x-p-q\right) + C_1\cot\left(\sqrt{C_1}x-p-q\right)\csc\left(\sqrt{C_1}x-p-q\right),\end{aligned} \tag{21}$$

where $p = \int -\frac{F_1(y)}{\sqrt{C_1}}\mathrm{d}y + C_5$ and $q = \int -(-5C_1 + 3C_2C_1 + 3F_2(t))\sqrt{C_1}\mathrm{d}t + C_6$.

So from Case 1, we find the Jacobi elliptic function, degenerative soliton-like and triangular function solutions of (1). We can also derive some other solutions if we make use of other cases.

Type 3.3. If we choose $l_1 = -k^2$, $m_1 = (2k^2-1)$, $n_1 = 1-k^2$, $l_2 = -1$, $m_2 = (2-k^2)$, $n_2 = k^2-1$, from (4), we derive $f = \mathrm{cn}(\alpha x+\xi)$, $g = \mathrm{dn}(\alpha x+\xi)$.

Thus we get the following Jacobi elliptic function solutions of (1):

$$\left.\begin{aligned} u &= a_0 + a_1\mathrm{cn}(\alpha x+\xi) + b_1\mathrm{dn}(\alpha x+\xi) + a_2\mathrm{cn}^2(\alpha x+\xi) + b_2\mathrm{cn}(\alpha x+\xi)\mathrm{dn}(\alpha x+\xi),\\ v &= c_0 + c_1\mathrm{cn}(\alpha x+\xi) + d_1\mathrm{dn}(\alpha x+\xi) + c_2\mathrm{cn}^2(\alpha x+\xi) + d_2\mathrm{cn}(\alpha x+\xi)\mathrm{dn}(\alpha x+\xi),\end{aligned}\right\} \tag{22}$$

where $\xi = p+q$ and α, p, q, a_0, a_1, b_1, c_0, c_1, d_1, a_2, b_2, c_2, d_2 satisfy (11)–(14), respectively.

For example, if the functions satisfy (11), namely Case 1, we derive the following Jacobi elliptic function solution of (1):

$$\begin{aligned} u_{31} &= C_2F_1(y) - \sqrt{k^2}F_1(y)\mathrm{cn}^2\left(-\frac{\sqrt{k^2}\sqrt{k^2C_1}x}{k^2}+p+q\right)\\ &\quad + F_1(y)\mathrm{cn}\left(-\frac{\sqrt{k^2\sqrt{k^2C_1}}x}{k^2}+p+q\right)\mathrm{dn}\left(-\frac{\sqrt{k^2\sqrt{k^2C_1}}x}{k^2}+p+q\right),\\ v_{31} &= F_2(t) - \sqrt{k^2}C_1\mathrm{cn}^2\left(-\frac{\sqrt{k^2\sqrt{k^2}C_1x}}{k^2}+p+q\right)\\ &\quad + C_1\mathrm{cn}\left(-\frac{\sqrt{k^2\sqrt{k^2}C_1x}}{k^2}+p+q\right)\mathrm{dn}\left(-\frac{\sqrt{k^2\sqrt{k^2}C_1x}}{k^2}+p+q\right),\end{aligned} \tag{23}$$

where $p = \int -\frac{F_1(y)\sqrt{k^2\sqrt{k^2C_1}}}{k^2C_1}\mathrm{d}y + C_5$ and $q = \int \frac{\left(-\sqrt{k^2}C_1 + 5k^2\sqrt{k^2}C_1 - 3C_2C_1k^2 - 3k^2F_2(t)\right)\sqrt{k^2\sqrt{k^2}C_1}}{k^4}\mathrm{d}t + C_6$.

So from Case 1, we find the Jacobi elliptic function solution of (1). We can also derive some other solutions if we make use of other cases.

Type 3.4. If we choose $l_1 = 1-k^2$, $m_1 = (2k^2-1)$, $n_1 = -k^2$, $l_2 = (1-k^2)$, $m_2 = (2-k^2)$, $n_2 = 1$, from (4), we derive $f = \mathrm{nc}(\alpha x+\xi)$, $g = \mathrm{sc}(\alpha x+\xi)$.

So we get the following Jacobi elliptic function solutions of (4):

$$\left.\begin{aligned} u &= a_0 + a_1\mathrm{nc}(\alpha x+\xi) + b_1\mathrm{sc}(\alpha x+\xi) + a_2\mathrm{nc}^2(\alpha x+\xi) + b_2\mathrm{nc}(\alpha x+\xi)\mathrm{sc}(\alpha x+\xi), \\ v &= c_0 + c_1\mathrm{nc}(\alpha x+\xi) + d_1\mathrm{sc}(\alpha x+\xi) + c_2\mathrm{nc}^2(\alpha x+\xi) + d_2\mathrm{nc}(\alpha x+\xi)\mathrm{sc}(\alpha x+\xi), \end{aligned}\right\} \tag{24}$$

where $\xi = p+q$ and α, p, q, a_0, a_1, b_1, c_0, c_1, d_1, a_2, b_2, c_2, d_2 satisfy (11) – (14), respectively.

For example, if the functions satisfy (11), namely Case 1, we derive the following Jacobi elliptic function solution of (1):

$$\begin{aligned} u_{41} &= C_2F_1(y) + F_1(y)\mathrm{nc}^2\left(-\frac{\sqrt{(1-k^2)C_1}x}{1-k^2}+p+q\right) \\ &\quad + F_1(y)\mathrm{nc}\left(-\frac{\sqrt{(1-k^2)C_1}x}{1-k^2}+p+q\right)\mathrm{sc}\left(-\frac{\sqrt{(1-k^2)C_1}x}{1-k^2}+p+q\right), \\ v_{41} &= F_2(t) + C_1\mathrm{nc}^2\left(-\frac{\sqrt{(1-k^2)C_1}x}{1-k^2}+p+q\right) \\ &\quad + C_1\mathrm{nc}\left(-\frac{\sqrt{(1-k^2)C_1}x}{1-k^2}+p+q\right)\mathrm{sc}\left(-\frac{\sqrt{(1-k^2)C_1}x}{1-k^2}+p+q\right), \end{aligned} \tag{25}$$

where $p = \int -\frac{F_1(y)\sqrt{(1-k^2)C_1}}{(1-k^2)C_1}\mathrm{d}y + C_5$ and $q = -\frac{\sqrt{(1-k^2)C_1}\int 3C_2C_1k^2+5C_1k^2+3F_2(t)k^2-C_1-3C_2C_1-3F_2(t)\mathrm{d}t}{(-1+k^2)^2} + C_6$.

So from Case 1, we find the Jacobi elliptic function solution of (1). We can also find some other Jacobi elliptic function solutions if we make use of other cases. If $k \to 1$, or $k \to 0$, we derive other exact non-travelling wave solutions of (1).

Type 3.5. If we choose $l_1 = k^2$, $m_1 = -(1+k^2)$, $n_1 = 1$, $l_2 = -1$, $m_2 = (2-k^2)$, $n_2 = -(1-k^2)$, from (4), we derive $f = \mathrm{sn}(\alpha x+\xi)$, $g = \mathrm{dn}(\alpha x+\xi)$.

So we get the following Jacobi elliptic function solutions of (1):

$$\left.\begin{aligned} v &= c_0 + c_1\mathrm{sn}(\alpha x+\xi) + d_1\mathrm{dn}(\alpha x+\xi) + c_2\mathrm{sn}^2(\alpha x+\xi) + d_2\mathrm{sn}(\alpha x+\xi)\mathrm{dn}(\alpha x+\xi), \\ u &= a_0 + a_1\mathrm{sn}(\alpha x+\xi) + b_1\mathrm{dn}(\alpha x+\xi) + a_2\mathrm{sn}^2(\alpha x+\xi) + b_2\mathrm{sn}(\alpha x+\xi)\mathrm{dn}(\alpha x+\xi), \end{aligned}\right\} \tag{26}$$

where $\xi = p+q$ and α, p, q, a_0, a_1, b_1, c_0, c_1, d_1, a_2, b_2, c_2, d_2 satisfy (11) – (14), respectively.

For example, if the functions satisfy (11), namely Case 1, we derive the following Jacobi elliptic function solution of (1):

$$\begin{aligned} u_{51} &= C_2F_1(y) - \mathrm{i}kF_1(y)\mathrm{sn}^2\left(\frac{\sqrt{-\mathrm{i}C_1}x}{\sqrt{k}}+p+q\right) + F_1(y)\mathrm{sn}\left(\frac{\sqrt{-\mathrm{i}C_1}x}{\sqrt{k}}+p+q\right)\mathrm{dn}\left(\frac{\sqrt{-\mathrm{i}C_1}x}{\sqrt{k}}+p+q\right), \\ v_{51} &= F_2(t) - \mathrm{i}kC_1\mathrm{sn}^2\left(\frac{\sqrt{-\mathrm{i}C_1}x}{\sqrt{k}}+p+q\right) + C_1\mathrm{sn}\left(\frac{\sqrt{-\mathrm{i}C_1}x}{\sqrt{k}}+p+q\right)\mathrm{dn}\left(\frac{\sqrt{-\mathrm{i}C_1}x}{\sqrt{k}}+p+q\right), \end{aligned} \tag{27}$$

where $p = \frac{\sqrt{-\mathrm{i}C_1}\int F_1(y)\mathrm{d}y + C_5\sqrt{k}C_1}{\sqrt{k}C_1}$ and $q = \frac{-\sqrt{-\mathrm{i}C_1}\int \mathrm{i}C_1+4\mathrm{i}k^2C_1-3C_2C_1k-3kF_2(t)\mathrm{d}t+C_6k^{3/2}}{k^{3/2}}$.

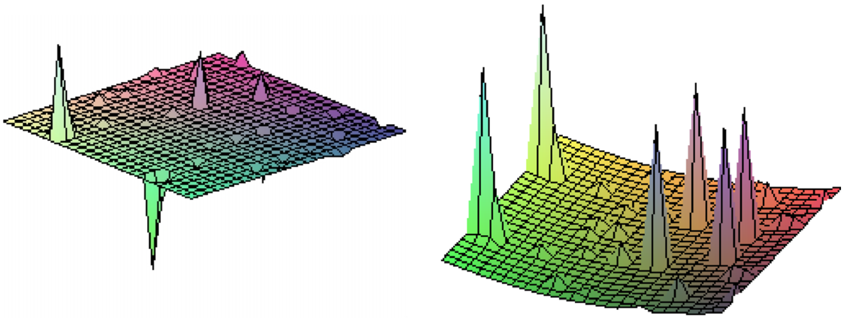

Fig. 1. Combined Jacobi elliptic function solution u_{21} (left) and v_{21} (right) of the $(2+1)$-dimensional KdV equation at $k = \frac{1}{750}$, $x = 0$, $C_1 = 4$, $F_1(y) = y$, $F_2(t) = t^2$, $C_2 = C_5 = C_6 = 1/2$.

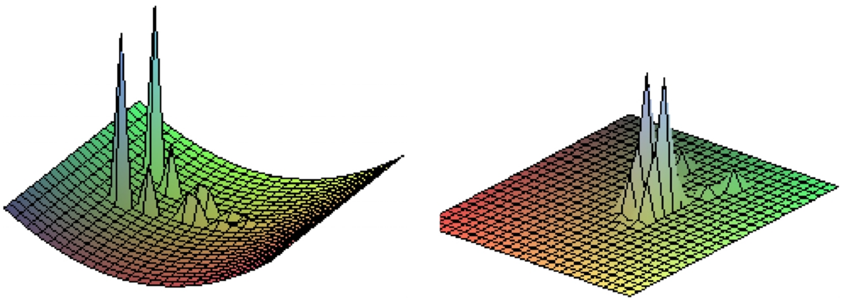

Fig. 2. Soliton-like solution u_{22} (left) and v_{22} (right) of the $(2+1)$-dimensional KdV equation at $x = 0, C_1 = 4, F_1(y) = y^2, F_2(t) = 2t, C_2 = C_5 = C_6 = 1/2$.

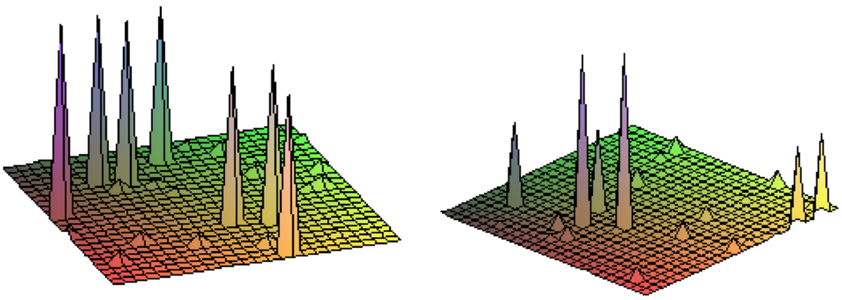

Fig. 3. Trigonometric function solution u_{23} (left) and v_{23} (right) of the $(2+1)$-dimensional KdV equation at $x = 0$, $F_1(y) = 2$, $F_2(t) = t$, $C_1 = C_2 = C_5 = C_6 = 1/2$.

So from Case 1, we find the Jacobi elliptic function solution of (1). We can also find some other Jacobi elliptic function solutions if we make use of other cases. If $k \to 1$, or $k \to 0$, we can derive other exact non-travelling wave solutions of (1).

Of course, there are other formal solutions of (1). They are omitted here for simplicity.

Remark 2. The properties of some new exact solutions of (1), as some illustrative samples, are shown in some figures. For example, the properties of solutions of Type 3.2 are shown in Figures 1 – 3.

Of course, we can also plot the other figures of the exact solutions of the $(2+1)$-dimensional KdV equation. We omit them here for convenience.

4. Conclusion and Discussion

In summary, based on Yan [21] and Liu and Yang [20], using symbolic computation, we have improved the extended F-expansion method in [20] and proposed the further improved F-expansion method to solve the $(2+1)$-dimentional KdV equation (1). Compared to the Jacobi elliptic function expansion method in [21 – 23] and F-expansion method [24 – 25], our method is more convenient and can get more new exact solutions. Although it is very complicated, we make use of the powerful symbolic computation system Maple, which makes the process simple. Among the solutions, the arbitrary functions imply that these solutions have rich local structures. The obtained solutions obtained may be of important significance for the explanation of some practical physical problems. In fact, this method is readily applicable to a large variety of nonlinear PDEs, such as the Perturbed-KdV equation, the $(2+1)$-dimensional Konopelchenko-Dubrovsky equation, the $(2+1)$-dimensional break-soliton equation.

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